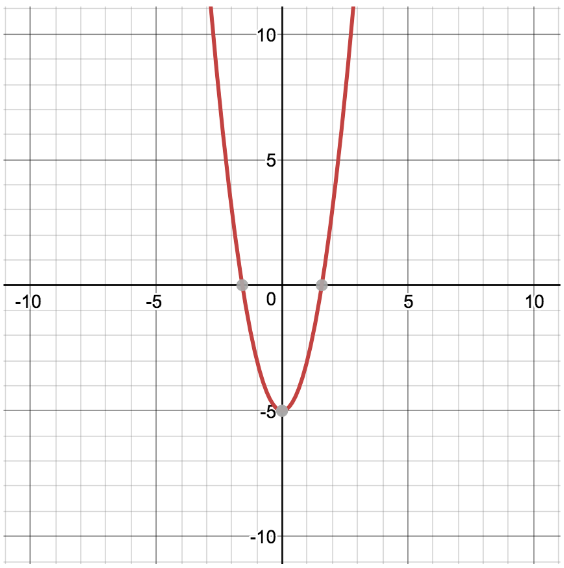
What is a limit?

A limit is the value that x approaches on a graph. There are many ways to determine the value of a limit or if a limit exists mathematically or by using a graph.

How can a limit be determined using a graph?

A limit can be easily determined by a graph unless the graph has a hole or jumps. For example, if you were to graph the function $f(x) = 2x^2 - 5$ to find the limit as $x \to 0$, it would look like this:



$$\lim_{x \to 0^+} \left(2x^2 - 5\right) = -5$$

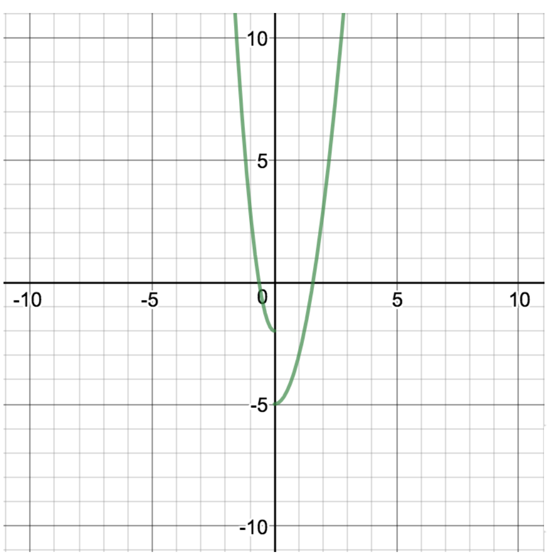
$$\lim_{x \to 0^-} \left(2x^2 - 5\right) = -5$$

↓

$$\lim_{x \to 0} \left(2x^2 - 5\right) = -5$$

Since x approaches 0 from the right and from the left, the limit must be -5.

However, some graphs may jump as x approaches a certain point. For example, if you were to graph the piecewise function $f(x) = \begin{cases} 2x^2 - 5, & x > 0 \\ 5x^2 - 2, & x \leq 0 \end{cases}$ to find the the limit as $x \to 0$, it would look like this:



$$\lim_{x \to 0^+} \left(f(x)\right) = -5$$

$$\lim_{x \to 0^-} \left(f(x)\right) = -2$$

↓

$$\lim_{x \to 0} \left(f(x)\right) = \textit{does not exist}$$

Since the limit as x approaches 0 from the left and from the right are both different values, the limit cannot exist.

How can a limit be determined mathematically?

Sometimes, there is not enough time to graph a function, or it is just easier to do the math to find the limit of a function. There are several ways to determine the limit of a function mathematically.

☆ If one method determines that the limit is undefined, use all the other methods listed to ☆
☆ confirm that the limit is actually undefined ☆

- **Plugging in for x**

Plug in the value that x is approaching to determine the limit.

$$\lim_{x\to 0}\left(2x^2 - 5\right) =$$
$$= 2(0)^2 - 5$$
$$= 0 - 5$$
$$= -5$$

If you plug in the value that x is approaching and get a non-zero number in the numerator and zero in the denominator, then you probably have found an asymptote. If the numerator is positive, then the limit is +∞, and if the numerator is negative, then the limit is -∞.

- **Factoring**

Factor the top and/or bottom of a fraction and cancel appropriate values so that the x value can be plugged in.

$$\lim_{x\to -1}\left(\frac{x^2+4x+3}{x^2-2x-3}\right) =$$
$$= \frac{(x+3)(x+1)}{(x-3)(x+1)}$$
$$= \frac{(x+3)\cancel{(x+1)}}{(x-3)\cancel{(x+1)}}$$
$$= \frac{x+3}{x-3}$$

$$= \frac{(-1)+3}{(-1)-3}$$

$$= -\frac{2}{4}$$

- **Conjugates**

Find the conjugate of the top or bottom of a fraction and cancel appropriate values so that the x value can be plugged in.

$$\lim_{x \to 4}\left(\frac{\sqrt{x}-2}{x-4}\right) =$$

$$= \left(\frac{\sqrt{x}-2}{x-4}\right)\left(\frac{\sqrt{x}+2}{\sqrt{x}+2}\right)$$

$$= \left(\frac{\cancel{x-4}}{\cancel{x-4}}\right)\left(\frac{1}{\sqrt{x}+2}\right)$$

$$= \frac{1}{\sqrt{x}+2}$$

$$= \frac{1}{\sqrt{(4)}+2}$$

$$= \frac{1}{2+2}$$

$$= \frac{1}{4}$$

- **Trigonometry Identities**

If a function involves trig identities, you may have to rewrite the equation to find the limit.

$$\lim_{x\to 0}\left(\frac{\sin(x)}{\sin(2x)}\right) =$$

$$= \frac{\sin(x)}{2\sin(x)\cos(x)}$$

$$= \frac{\cancel{\sin(x)}}{2\cancel{\sin(x)}\cos(x)}$$

$$= \frac{1}{2\cos(x)}$$

$$= \frac{1}{2\cos(0)}$$

$$= \frac{1}{2(1)}$$

$$= \frac{1}{2}$$

- **L'hopital's Rule**

If you plug in zero to a function and you get 0/0 or ±∞/±∞, you will have to use L'hopital's Rule. You will take the derivative of top and bottom of a fraction to find the limit. This rule works when x is approaching any real number, positive infinity, or negative infinity.

$$\lim_{x\to 2}\left(\frac{x-2}{x^2-4}\right) =$$

$$f'(x) = \frac{\frac{(x+h)-2-(x-2)}{h}}{\frac{(x+h)^2-4-\left(x^2-4\right)}{h}}$$

$$f'(x) = \frac{\frac{x + h - 2 - x + 2}{h}}{\frac{x^2 + 2hx + h^2 - 4 - x^2 + 4}{h}}$$

$$f'(x) = \frac{\frac{\cancel{x} + h - \cancel{2} - \cancel{x} + \cancel{2}}{h}}{\frac{\cancel{x^2} + 2hx + h^2 - \cancel{4} - \cancel{x^2} + \cancel{4}}{h}}$$

$$f'(x) = \frac{\frac{\cancel{h}}{\cancel{h}}}{\frac{2x\cancel{h} + h^{\cancel{2}}}{\cancel{h}}}$$

$$f'(x) = \frac{1}{2x}$$

$$f'(x) = \frac{1}{2(2)}$$

$$f'(x) = \frac{1}{4}$$